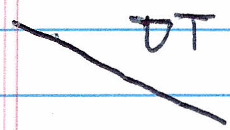
Transport I

$$\left\{\begin{array}{l} \dfrac{\langle E^2\rangle_{\omega}}{8\pi} = \dfrac{T}{\omega}\,\dfrac{\mathrm{Im}\,\epsilon}{|\epsilon|^2} \\ \text{shape } \langle f\rangle \to \text{flctns.} \end{array}\right.$$

→ Can observe:

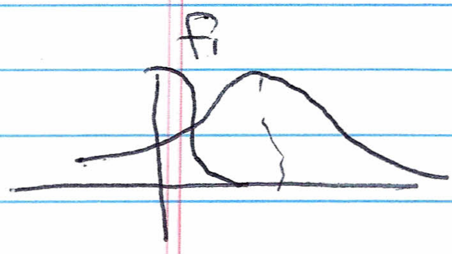
- stable plasma will tend to relax, by return to Maxwellian (global)

i.e.



$\Rightarrow Q = -\chi \nabla T$

↓ collisional diffusion



$\Rightarrow \underline{E} = \eta \underline{J}$

$= \eta(-n_0 e\, \underline{u_0})$

- process: collisions → i.e. Coulomb scattering for plasma

- stages: 1 τ_c ⇒ local Maxwellian
  many τ_c ⇒ relaxation

- relaxation ⟺ Entropy Production

i.e. $\dfrac{dS}{dt} \simeq -Q\,\nabla T$

Q ↓ flux, ∇T ↳ force.

→ Theory :

(I)
- Boltzmann Eqn. and H-Thm.

  ⇓

- Landau Eqn (Boltzmann Eqn. for glancing Coulomb collision)

  ⇓

- Balescu-Lenard Eqn. (Screened Landau Eqn.)

  or

(II) - Fokker-Planck Eqn.

  ⇓ Coulomb Collisions (glancing)

- Landau Eqn.

  ⇓ convenient/clever

- Rosenbluth potentials.

(I) For plasma,

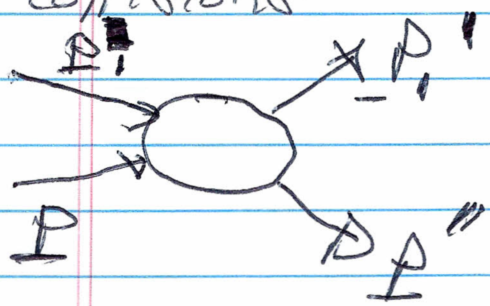
$$\frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f + \frac{q}{m}\left(\underline{E} + \cancel{\frac{\underline{v}\times\underline{B}}{c}}\right)\cdot\frac{\partial f}{\partial \underline{v}} = C(f)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f + \frac{q}{m}\underline{E}\cdot\frac{\partial f}{\partial \underline{v}} = C(f)$$

~ Phase space density no longer conserved

~ Origin of C:

- collisions



$\underline{p}$ ≡ 'test' particle

$\underline{p}_1$ ≡ 'background' or "field" particle

⇓

binary collisions,
(N.B. what is 'a collision' in sense of TPM)

$W(\underline{p}, \underline{p}_1; \underline{p}_1', \underline{p}')$ → transition probability

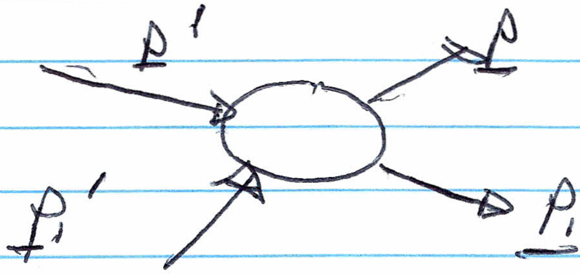
or

- correlations ⇒ residue of BBGKY hierarchy

Convenient to view Boltzmann Eqn. as scattering into/out of state $\underline{p}$:

$$\frac{df}{dt} = C(\underline{p}) = \text{rate } \underset{\sim}{\text{in}} - \text{rate } \underset{\sim}{\text{out}}$$

in:

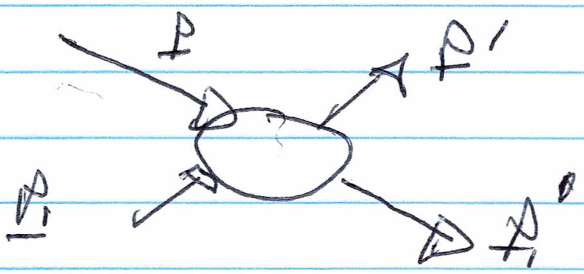


So

$$\text{in} = \int d\underline{p}' \int d\underline{p}_1' \int d\underline{p}_1 \; W(\underline{p}', \underline{p}_1'; \underline{p}, \underline{p}_1) \, * \, f(\underline{p}') f(\underline{p}_1')$$

and:

out:



$$\text{out} = \int d\underline{p}_1 \int d\underline{p}' \int d\underline{p}_1' \; f(\underline{p}) f(\underline{p}_1) \, W(\underline{p}, \underline{p}_1; \underline{p}', \underline{p}_1')$$

So

$$\boxed{\frac{d f(\underline{p})}{dt} = \int d\underline{p}_1 \int d\underline{p}' \int d\underline{p}_1' \; W(\underline{p}, \underline{p}_1; \underline{p}', \underline{p}_1') \, * \, \left(f(\underline{p}') f(\underline{p}_1') - f(\underline{p}) f(\underline{p}_1) \right)}$$

→ Boltzmann Equation.

Content:

① → transition probability : contains cross-section, other micro-physics

② ⇒ $f(\underline{p})\, f(\underline{p}_1)$ → used "Principle of Molecular Chaos" (Boltzmann) to simplify

$$f(\underline{p}, \underline{p}_1) = f(\underline{p})\, f(\underline{p}_1)$$ → $\left\{ \begin{array}{l} \text{probabilities} \\ \text{independent} \end{array} \right.$

⇒ diluteness, again

③ ⇒

$$w(\underline{p}, \underline{p}_1 ; \underline{p}', \underline{p}_1') = w(\underline{p}', \underline{p}_1' ; \underline{p}, \underline{p}_1)$$

"Principle of detailed Balance" ⇒ Probability of Forward Transition = Probability of Back Transition

Further observe:

- one integration trivial, as

$$\underline{p} + \underline{p}_1 = \underline{p}' + \underline{p}_1'$$

- $C(f) = 0$ for $f = f_0$

i.e. $f_0 = c \exp\left[-\frac{(\varepsilon + p \cdot V)}{T} \right]$

$C \to 0$, due conservation energy and momentum.

- will show f_0 renders $dS/dt = 0$.

- for $C(f)$ number conserving, must have:

$C(f) = -\underline{\partial}_p \cdot \underline{J}(p) = -\partial \underline{J} / \partial \underline{p}$

i.e. $C(f)$ as divergence of flux in momentum space.

This brings us to:

$\boxed{H\text{-Theorem}}$

- a gas / plasma, left alone will evolve to an equilibrium of maximal entropy

- evolution accompanied by entropy production.

- evolution is to uniform Maxwellian
- $dS/dt \geq 0$

for ideal gas

$$S = \int d\underline{x} \int d\underline{p}\; f \ln(e/f)$$

$$\simeq \int dx \int dp \left[-f \ln f \right]$$

see notes on entropy, next lecture.

Will show $dS/dt \geq 0$.

n.b. Vlasov Eqn. cons. entropy

$$\frac{dS}{dt} = -\int d\Gamma \left[\frac{df}{dt} \ln f + \cancel{f} \frac{1}{\cancel{f}} \frac{df}{dt} \right]$$

$$= -\int d\Gamma \left[C(f) \ln(f) + C(f) \right]$$

$$= -\int d\underline{x} \int d\underline{p}\; \ln(f)\, C(f)$$

→ entropy production due explicitly to collisions

$$= -\int dx \int d\underline{p} \int d\underline{p}_1 \int d\underline{p}' \int d\underline{p}_1'\; \ln f\; w \left(f(\underline{p}') f(\underline{p}_1') - f(\underline{p}) f(\underline{p}_1) \right)$$

Lemma

$$\int \varphi(\underline{p})\, C(\underline{p})\, d\underline{p}$$

$$= \tfrac{1}{2}\int d^4\underline{p}\, \left(\varphi + \varphi_1 - \varphi' - \varphi_1'\right) w f' f_1'$$

where notational shorthand $\ni$

$$d^4\underline{p} = d\underline{p}\, d\underline{p}_1\, d\underline{p}_1'\, d\underline{p}'$$

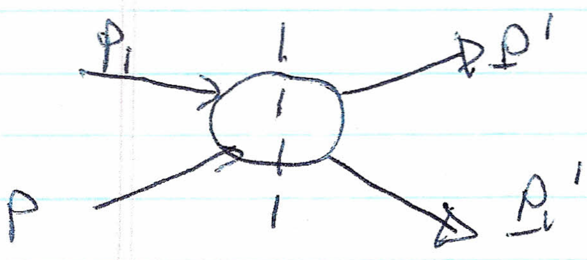
Now, explicitly:

$$\int d\underline{p}\, \varphi(\underline{p})\, C(\underline{p}) = \int \varphi\, \overset{①}{w}(\underline{p}, \underline{p}_1; \underline{p}', \underline{p}_1')\, f' f_1'\, d^4p$$

$$- \int \varphi\, \overset{②}{w}(\underline{p}', \underline{p}_1'; \underline{p}, \underline{p}_1)\, f f_1$$

now, in ②:

$\rightsquigarrow$ interchange $\underline{p}, \underline{p}_1 \leftrightarrow \underline{p}', \underline{p}_1'$



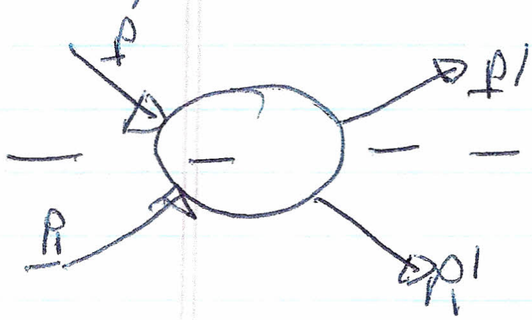
flip } about ! rotate }

use T symmetry.

so

$$\int d\underline{p}\,\varphi\,C(f) = \int d^4p \left\{ \left(\varphi(\underline{p}) - \varphi(\underline{p}')\right) w(\underline{p},\underline{p}_1;\underline{p}',\underline{p}_1') \times f' f_1' \right\}$$

Now, consider:



and interchange about — — —

i.e. $\underline{p}, \underline{p}'$ with $\underline{p}_1, \underline{p}_1'$

n.b. up-down symmetry equivalent

so

$\int d\underline{p}\, C(f)\,\varphi =$ [redacted]

$$= \frac{1}{2} \int d^4p \left\{ \left(\varphi(\underline{p}) - \varphi(\underline{p}') + \varphi(\underline{p}_1) - \varphi(\underline{p}_1') \right) \times w\, f' f_1' \right\}$$

this proves Lemma!

Now, let $\varphi = \ln f$,

so Lemma $\Rightarrow$

$$\frac{dS}{dt} = -\frac{1}{2}\int d\underline{x} \int d^4\underline{p} \, \big(\ln f + \ln f_1 - \ln f' - \ln f_1'\big) * w \, f' f_1'$$

$$= \frac{1}{2}\int d\underline{x} \int d^4\underline{p} \; w \, f' f_1' \ln\left(f' f_1' / f f_1\right)$$

$x \equiv f' f_1' / f f_1$

$$\boxed{\frac{dS}{dt} = \frac{1}{2}\int d\underline{x} \int d^4\underline{p} \; w \, f f_1 \, x \ln x}$$

Now since $\int c(f)\, d\Gamma = 0$

have $\int w f f_1 \, (x-1)\, d^4\underline{p}\, d\underline{x} \equiv 0$

i.e. write zero in complex way.

so adding:

$$\boxed{\frac{dS}{dt} = \frac{1}{2}\int d^4p \int dx \; w f f_1 \left[x \ln x - x + 1\right]}$$

gives entropy production rate.

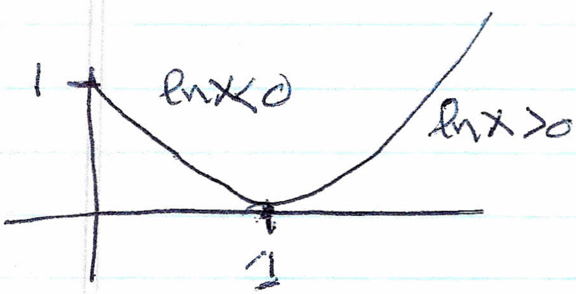
$F(x) = x \ln x - x + 1$

$F' = 1 + \ln x - 1$

$F(0) = 1$

$F(1) = 0$

so



so

$$\boxed{dS/dt \geq 0}$$

Boltzman H-thm!

- $dS/dt = 0$ for $x = 1$

$f f_1 = f' f_1'$

$\ln f + \ln f_1 = \ln f' + \ln f_1'$

$\Rightarrow \quad \ln f + \ln f_1 = \text{const.}$

sum of logs conserved in collision

$\Rightarrow \ln f = c + \underline{\beta} \cdot \underline{V} + \alpha E$

$\alpha < 0$

{ see next lecture

$\frac{ds}{dt} = 0$ determines Maxwellian

→ key: → detailed balance ⟺ w symmetry

→ molec. chaos
$f(1,2) = f(1) f(2)$

→ $ds/dt \geq 0$

$ds/dt = 0$ corresponds $C(f) = 0$

collisions drive system to equilibrium

→ $d\underline{x}$ irrelevant !!!

entropy produced locally

i.e. relaxation to local Maxwellian.

→ Essence of H-thm. is:

macroscopic irreversibility from microscopically reversible dynamics + molec. chaos (micro-chaos)

From Boltzmann $\rightarrow$ Landau:

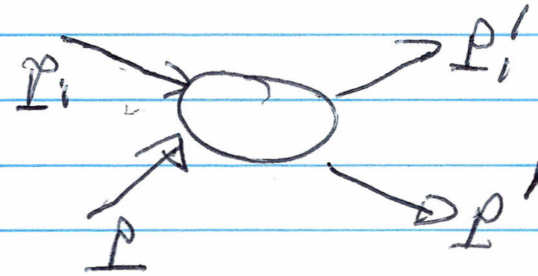
$$\frac{df}{dt} = C(f)$$

$$= \int d\underline{p}_1 \int d\underline{p}' \int d\underline{p}_1' \, W \left(\overset{①}{f(\underline{p}')f(\underline{p}_1')} - \overset{②}{f(\underline{p})f(\underline{p}_1)} \right)$$

$$\underline{p} + \underline{p}_1 = \underline{p}' + \underline{p}_1'$$

and

$$C(f) = -\frac{\partial}{\partial \underline{p}} \cdot \underline{J}(f)$$



Key point for Coulomb scattering:

$$\boxed{\text{scattering is event of } \underline{\text{small}} \text{ momentum transfer}}$$

i.e. can proceed by small momentum transfer limit of Boltzmann Eqn.

i.e.

$$\underline{p} \rightarrow \underline{p} + \underline{q}/2$$

$$\underline{p}' \rightarrow \underline{p}' - \underline{q}/2$$

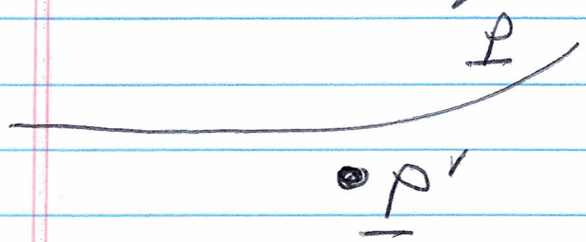
Consider terms in B.E.:

i.e. term (2), with $p_1 \to p'$ (re-label)

$\Rightarrow$ # collisions per time between particle with momentum p (test) and particle with momentum p'.

$$\# = W(p+q/2,\, p'-q/2,\, q)\,\langle f(p)\, f(p')\rangle\, d^3p'\, d^3q$$

i.e.



$p \to$ test particle, scatter-ee

$p' \to$ background or field particle, scatter-er.

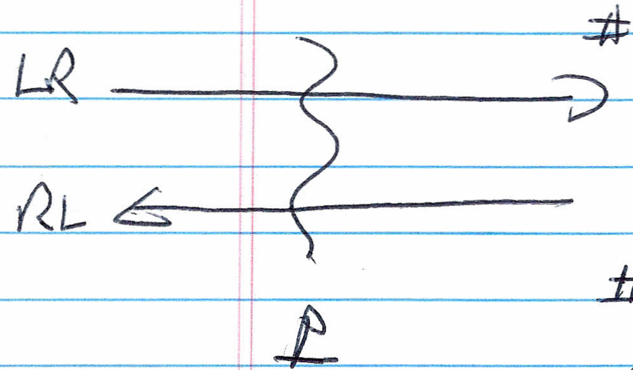
$q \equiv$ momentum transfer.

$$C(f) = -\frac{\partial}{\partial p}\cdot J(p)$$

easier to construct $J(p)$, scattering current

So

$\underline{J}(\underline{p})$:



Left → Rt
i.e. upshift in momentum

Rt → left
i.e. decrease in momentum

$$J(p) \to \underset{\text{mom transfer}}{LR} + \underset{\text{bckgnd}}{RL}$$

$$\# LR = \sum_{\text{spec}}^{ee,\,ei} \int_{|q| \geq 0} d^3\underline{q} \int d^3 p' \int_{\underline{p}-\underline{q}}^{\underline{p}} \Big[W\left(\underline{q};\, p + \tfrac{q}{2},\, p' - \tfrac{q}{2}\right) \times \langle f(p) \rangle \langle f(p') \rangle \Big]$$

$$\# RL = \sum_{\text{spec}}^{ee,\,ei} \int_{|q| > 0} d^3 q \int d^3 p' \Big[W\left(-\underline{q},\, p + \tfrac{q}{2},\, p' - \tfrac{q}{2}\right) \times \langle f\left(p + \tfrac{q}{2}\right) \rangle \langle f\left(p' - \tfrac{q}{2}\right) \rangle$$

Now

— detailed balance: $w(\underline{q}) = w(-\underline{q})$

— small deflection:

$$w(\underline{p}+\underline{q}/2,\ \underline{p}'-\underline{q}/2;\ \underline{q}) \cong w(\underline{p},\underline{p}';\underline{q})$$

so, for small $\underline{q}$:

$$\underline{J}(\underline{p}) = \sum_{\text{spec.}} \int_{|\underline{q}|>0} d^3q \int d^3p' \int_{\underline{p}-\underline{q}}^{\underline{p}} w \Big[\langle f(\underline{p})\rangle \langle f(\underline{p}')\rangle - \langle f(\underline{p}+\underline{q})\rangle \langle f(\underline{p}'-\underline{q})\rangle \Big]$$

expand

$$= [\quad] \Big\{ \cancel{\langle f(\underline{p})\rangle \langle f(\underline{p}')\rangle} - \cancel{\langle f(\underline{p})\rangle \langle f(\underline{p}')\rangle} + \underline{q}\cdot\frac{\partial\langle f(\underline{p}')\rangle}{\partial \underline{p}'}\langle f(\underline{p})\rangle - \underline{q}\cdot\frac{\partial\langle f\rangle}{\partial \underline{p}}\langle f(\underline{p}')\rangle \Big\}$$

N.B.: Unsurprisingly, $\langle f\rangle$ gradients set current;

and,

$$[\;\;] = (\;)\int_{p_\alpha - q_\alpha}^{p_\alpha} \equiv (\;)\, q_\alpha$$

So

$$\boxed{J(p) = \sum_s \int d^3p' \left[\langle f(p)\rangle \frac{\partial}{\partial p'_\beta} \langle f(p')\rangle - \langle f(p')\rangle \frac{\partial}{\partial p_\beta} \langle f(p)\rangle \right] B_{\alpha\beta}}$$

$$B_{\alpha,\beta} = \int d^3q\, \frac{W(q)}{2}\, q_\alpha q_\beta \rightarrow \text{scattering vector} \sim \langle q^2 \rangle$$

↳ prevents dbl counting

But

$$W d^3q = |v - v'|\, d\sigma$$

$$= m|q|\, d\sigma$$

So

$$B_{\alpha\beta} = \int \frac{d^3q}{2}\, |v - v'|\, q_\alpha q_\beta\, d\sigma.$$

so

$$\frac{df}{dt} = -\frac{\partial}{\partial \underline{p}} \cdot \underline{J}(\underline{p})$$

$$\underline{J}(\underline{p}) = \left[-\frac{\partial}{\partial \underline{p}} \cdot \underline{\underline{D}} \langle f(\underline{p}) \rangle + \underline{F} \langle f(\underline{p}) \rangle \right]$$

$\partial/\partial \underline{p}$ pulls thru

$$\underline{\underline{D}} = \sum_{s} \int d^3p' \langle f(\underline{p}') \rangle \underline{\underline{B}}_{\alpha,\beta}$$

→ diffusion tensor (in velocity)

→ scattering by "field" particles ($\langle f(\underline{p}') \rangle$)
i.e. test particle diffused by background particles

$$\underline{F} = \sum_{s} \int d^3p' \frac{\partial \langle f(\underline{p}') \rangle}{\partial p'_{\beta}} B_{\alpha,\beta}$$

→ drag, dynamical friction

→ friction exerted on test particle by field particles.

N.B.

$$\frac{d f}{dt} = -\frac{\partial}{\partial \underline{p}} \cdot \underline{\Gamma}(\underline{p})$$

then for "slowing down":

$$\frac{d}{dt}\langle \underline{p} \rangle = \frac{d}{dt}\int d\underline{p}\, \underline{p}\, f$$

$$= \int d\underline{p} \left[\underline{F} \langle f(\underline{p}) \rangle - \cancel{\frac{\partial}{\partial \underline{p}} \cdot \underline{\underline{D}} \langle f(\underline{p}) \rangle} \right]$$

$$= \int d\underline{p} \left[\underline{F} \langle f(\underline{p}) \rangle \right]$$

controls slowing.

and for "energization":

$$\frac{d}{dt}\left\langle \frac{p^2}{2} \right\rangle = \int d\underline{p} \left[\underline{\underline{D}} \langle f(\underline{p}) \rangle - \underline{p} \cdot \underline{F} \langle f(\underline{p}) \rangle \right]$$

increases mean square p.

In Fokker-Planck Formulation:

$$\underline{\underline{D}} = \langle \Delta \underline{p}\, \Delta \underline{p} \rangle / 2\Delta t$$

$$\underline{F} = \left\langle \frac{\Delta \underline{p}}{\Delta t} \right\rangle$$

To simplify:

- $|\underline{q}|$ small, small angle scattering.

$\underline{q} \perp \underline{V} - \underline{V}'$

deflection

so

- $(V_\beta - V'_\beta) B_{\alpha,\beta} = 0$

i.e. $\underline{\underline{B}}$ transverse to $\underline{v} - \underline{v}'$ and depends on $\underline{v} - \underline{v}'$,

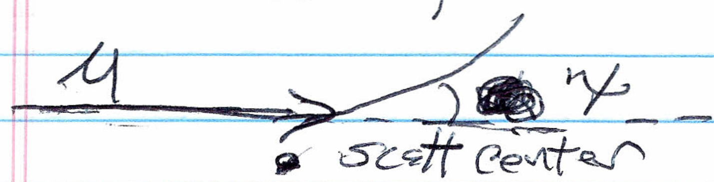
$$B_{\alpha,\beta} = B \left[\delta_{\alpha,\beta} - \frac{(V_\alpha - V'_\alpha)(V_\beta - V'_\beta)}{(V - V')^2} \right]$$

$$B = \frac{1}{2} \int \underline{q}^{\prime 2} |v - v'| \, d\sigma = B_{\alpha,\alpha}.$$

For $B_{\alpha,\beta}$:

χ = angle deviation $\underline{v}-\underline{v}'$ (upon collision)

recall



$|q| = \mu\, |v-v'|\, \chi$ $\qquad (\sin\chi \sim \chi)$

reduced mass

$\Rightarrow$

$$B = \frac{1}{2}\int d\sigma\, q^2\, |v-v'|$$

$$= \frac{1}{2}\int d\sigma\, \left(\mu^2 |v-v'|^2 \chi^2\right) |v-v'|$$

$$= \frac{\mu^2}{2}\, |v-v'|^3 \int \chi^2\, d\sigma$$

but $\sigma_T = \int (1-\cos\chi)\, d\sigma$ → {transverse transport cross section}

transverse scattering

$$\cong \int d\sigma\, \frac{\chi^2}{2}$$

Now, for Coulomb scattering:

$$d\sigma = 4(ee')^2 d\chi / \mu^2 v_{rel}^4 \chi^3$$

so

$$B = \frac{\cancel{\mu^2}}{\cancel{2}} \cancel{|v-v'|^2} \int \frac{d\chi}{\cancel{2}} \frac{\cancel{4}(ee')^2 \cancel{\chi^2}}{\cancel{\mu^2} \cancel{v_{rel}^4}_{v_{rel}^{\cancel{2}}} \cancel{\chi^3}} \cancel{v_{rel}}$$

$$= \frac{1}{|v_{rel}|^{\cancel{2}}} (ee')^2 \int \frac{d\chi}{\chi}$$

$$= \frac{(ee')^2}{|v_{rel}|} \int \frac{d\chi}{\chi}$$

Finally,

$$\left\{ \begin{aligned} B_{\alpha,\beta} &= B\left[\delta_{\alpha,\beta} - \frac{(v_\alpha - v'_\alpha)(v_\beta - v'_\beta)}{(v-v')^2} \right] \\ B &= \frac{(ee')^2}{|v_{rel}|} \ln \Lambda \end{aligned} \right.$$

→ Coulomb logarithm.

and

$$\begin{cases} C(f)_{\text{Landau}} = -\dfrac{\partial}{\partial \underline{p}} \cdot \underline{J}(\underline{p}) \\ \underline{J}(\underline{p}) = \left[-\dfrac{\partial}{\partial \underline{p}} \cdot \underline{\underline{D}} \langle f(\underline{p}) \rangle + \underline{F} \langle f(\underline{p}) \rangle \right] \end{cases}$$

$$\begin{cases} \underline{\underline{D}} = \sum\limits_{s} \int d^3p' \langle f(\underline{p}') \rangle \underline{\underline{B}}_{\alpha,\beta} \\ \underline{F} = \sum\limits_{s} \int d^3p' \dfrac{\partial}{\partial p'_{\beta}} \langle f(\underline{p}') \rangle B_{\alpha,\beta} \end{cases}$$

completes Landau Collision operator